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Abstract

The Data Correcting Algorithm is a branch and bound algorithm in which the data of a given problem instance is 'corrected' at each branching in such a way that the new instance will be as close as possible to a polynomially solvable instance and the result satisfies an acceptable accuracy (the difference between optimal and current solution). In this paper the data correcting algorithm is applied to determining exact and approximate optimal solutions to the simple plant location problem. Implementations of the algorithm are based on a pseudo-Boolean representation of the goal function of the SPLP and a new reduction rule. We study the efficiency of the data correcting approach using two different bounds, the combinatorial bound and the Erlenkotter bound. We present computational results on several benchmark instances of the simple plant location problem, which confirm the efficiency of the data-correcting approach.

Keywords: Simple Plant Location Problem, Pseudo-Boolean representation, Beresnev function, Branch and Bound, Preprocessing.

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1. Introduction

The Simple Plant Location Problem (SPLP) takes a set $I = \{1, 2, \ldots, m\}$ of sites in which plants can be located, a set $J = \{1, 2, \ldots, n\}$ of clients, each having a unit demand, a vector $F = (f_i)$ of fixed costs for setting up plants at sites $i \in I$, and a matrix $C = [c_{ij}]$ of transportation costs from $i \in I$ to $j \in J$ as input. It computes a set $P^\star$, $\emptyset \subset P^\star \subseteq I$, at which plants can be located so that the total cost of satisfying all client demands is minimal. The costs involved in meeting the client demands include the fixed costs of setting up plants, and the transportation cost of supplying clients from the plants that are set up. A detailed introduction to this problem has appeared in Cornuejols *et al.* [14]. The SPLP forms the underlying model in several combinatorial problems, like set covering, set partitioning, information retrieval, simplification of logical Boolean expressions, airline crew scheduling, vehicle despatching (Christofides [10]), assortment (Beresnev *et al.* [6], Goldengorin [19], Jones *et al.* [25], Pentico [32,33], Tripathy *et al.* [36]), and is a subproblem for various location analysis problems (Revelle and Laporte [34]).

The SPLP is $\mathcal{NP}$-hard (Cornuejols *et al.* [14]), and several exact and heuristic algorithms for solving it have been discussed in the literature. Most of the exact algorithms are based on a mathematical programming formulation of the SPLP (see for example, Schrage [35], Morris [30], Held *et al.* [24], Cornuejols *et al.* [12], Cornuejols and Thizy [13], and Garfinkel *et al.* [17]). Polyhedral results for the SPLP polytope have been reported in Trubin [37], Balas and Padberg [1], Mukendi [31], Cornuejols *et al.* [11], Krarup and Pruzan [28], Cho *et al.* [8], and Cho *et al.* [9]. In theory, these results allow us to solve the SPLP by applying the simplex algorithm to the strong linear programming relaxation, with the additional stipulation that a pivot to a new extreme point is allowed only when this new extreme point is integral. However, efficient implementations of this pivot rule are not available. Beasley [2] reported computational experiments with Lagrangian heuristics for SPLP instances. Körkel [27] proposed algorithms based on refinements to a dual-ascent heuristic procedure to solve the dual of a linear programming relaxation of the SPLP (Körkel [27]), combined with the use of the complementary slackness conditions to construct primal solutions (Erlenkotter [15]). An annotated bibliography is available in Labbé and Louveaux [29]. An exact algorithm based on a pseudo-Boolean representation of the problem has been reported in Goldengorin *et al.* [22]. This algorithm uses a preprocessing rule to reduce the size of its input. The preprocessing rule is due to Khumawala [26].

It is common knowledge that exact algorithms for $\mathcal{NP}$-hard problems in general, and for the SPLP in particular, spend only about 10% of the execution time to find an optimal solution. The remaining time is spent proving the optimality of the solution. In this paper, our aim is to reduce the amount of time spent proving the optimality of the solution obtained. We propose a data correcting algorithm for the SPLP that is designed to output solutions with a pre-specified *acceptable accuracy* α. This means that the difference between the cost of the solution output by the algorithm is no more than α more than the cost of an optimal solution. (Note that $\alpha = 0$ results in an exact algorithm for the SPLP, while $\alpha = \infty$ results in a fast greedy algorithm.) The objective function of the SPLP is supermodular (see Cornuejols *et al.* [14]) and so, the data correcting algorithm described in Goldengorin *et al.* [20] can be used to solve the SPLP. In fact, Goldengorin *et al.* [20] contains an example to that effect. However, it can be made much more efficient; for example, by using SPLP-specific bounds (used in Erlenkotter [15]) and preprocessing rules (used in Khumawala [26]). The algorithm described here uses a pseudo-Boolean representation of the SPLP, due originally to Beresnev [5]. It uses a new reduction procedure based on data correcting, which is stronger than the preprocessing rules used in Khumawala [26] to reduce the original instance to a smaller 'core' instance, and then solves it using a procedure based on preliminary preservation and data correcting (see Goldengorin *et al.* [20]). Computational experiments with this algorithm on benchmark instances of the SPLP are also described in the paper. We show how the use of preprocessing and bounds specific to the SPLP enhance the performance of the data-correcting algorithm. This algorithm is based on three concepts found in the literature, a pseudo-Boolean representation of the SPLP, data-correcting, and the preliminary preservation procedure. The next section of this paper therefore contains a brief exposure to these concepts. We describe the new algorithm in Section 3 and present the results of our computational experiments with it in Section 4. We finally conclude the paper in Section 5 with a summary of the work presented here and discussions.

2. Preliminaries from the Literature

In this section we describe a pseudo-Boolean representation of the SPLP that we use in our algorithm (Subsection 2.1), an introduction to data correcting (Subsection 2.2), and a description of the preliminary preservation procedure (Subsection 2.3). The last two are described using the SPLP as an example.

2.1 A Pseudo-Boolean Representation of the SPLP

The pseudo-Boolean approach to solving the SPLP (Hammer [23], Beresnev [5]) is a penalty-based approach that relies on the fact that any instance of the SPLP has an optimal solution in which each client is supplied by exactly one plant. This implies, that in an optimal solution, each client will be served fully by the plant located closest to it. Therefore, it is sufficient to determine the sites where plants are to be located, and then use a minimum weight matching algorithm to assign clients to plants.

An instance of the SPLP can be described by a m-vector $F = (f_i)$, and a $m \times n$ matrix $C = [c_{ij}]$; $m, n \geq 1$. We will use the $m \times (n+1)$ *augmented matrix* $[F|C]$ as a shorthand for describing an instance of the SPLP. The total cost $f_{[F|C]}(P)$ associated with a subset P of I consists of two components, namely the fixed costs $\sum_{i \in P} f_i$ and the transportation costs $\sum_{j \in J} \min\{c_{ij} | i \in P\}$; i.e.

$$f_{[F|C]}(P) = \sum_{i \in P} f_i + \sum_{j \in J} \min\{c_{ij} | i \in P\},$$

and the SPLP is the problem of finding

$$P^\star \in \arg\min\{f_{[F|C]}(P) : \emptyset \subset P \subseteq I\}. \tag{1}$$

In the remainder of this subsection we describe the pseudo-Boolean formulation of the SPLP due to Beresnev [5].

A $m \times n$ *ordering matrix* $\Pi = [\pi_{ij}]$ is a matrix each of whose columns $\Pi_j = (\pi_{1j}, \ldots, \pi_{mj})^T$ define a permutation of $1, \ldots, m$. Given a transportation matrix C, the set of all ordering matrices Π such that $c_{\pi_{1j}j} \leq c_{\pi_{2j}j} \leq \cdots \leq c_{\pi_{mj}j}$ for $j = 1, \ldots, n$, is denoted by $perm(C)$.

Defining

$$y_i = \begin{cases} 0 & \text{if } i \in P \\ 1 & \text{otherwise,} \end{cases} \quad \text{for each } i = 1, \ldots, m \tag{2}$$

we can indicate any solution P by a vector $\mathbf{y} = (y_1, y_2, \ldots, y_m)$. The fixed cost component of the total cost can be written as

$$\mathcal{F}_F(\mathbf{y}) = \sum_{i=1}^{m} f_i(1 - y_i). \tag{3}$$

Given a transportation cost matrix C, and an ordering matrix $\Pi \in perm(C)$, we can denote differences between the transportation costs for each $j \in J$ as

$$\begin{aligned} \Delta c[0, j] &= c_{\pi_{1j}j}, \quad \text{and} \\ \Delta c[l, j] &= c_{\pi_{(l+1)j}j} - c_{\pi_{lj}j}, \quad l = 1, \ldots, m-1. \end{aligned}$$

Note that $\Delta c[l, j] \geq 0$, even if the transportation cost matrix C contains negative entries. The transportation costs of supplying any client $j \in J$ from any open plant can be expressed in terms of the $\Delta c[\cdot, j]$ values. It is clear that we have to spend at least $\Delta c[0, j]$ in order to satisfy j's demand since this is the cheapest cost of satisfying j. If no plant is located at the site closest to j, i.e. $y_{\pi_{1j}} = 1$, we try to satisfy the demand from the next closest site. In that case, we spend an additional $\Delta c[1, j]$. Continuing in this manner, the transportation cost of supplying $j \in J$ is

$$\begin{aligned} \min\{c_{i,j} | i \in S\} &= \Delta c[0, j] + \Delta c[1, j] \cdot y_{\pi_{1j}} + \Delta c[2, j] \cdot y_{\pi_{1j}} \cdot y_{\pi_{2j}} \\ &\quad + \cdots + \Delta c[m-1, j] \cdot y_{\pi_{1j}} \cdots y_{\pi_{(m-1)j}} \\ &= \Delta c[0, j] + \sum_{k=1}^{m-1} \Delta c[k, j] \cdot \prod_{r=1}^{k} y_{\pi_{rj}}, \end{aligned}$$

so that the transportation cost component of the cost of a solution $\mathbf{y}$ corresponding to an ordering matrix $\Pi \in perm(C)$ is

$$\mathcal{T}_{C,\Pi}(\mathbf{y}) = \sum_{j=1}^{n} \left\{ \Delta c[0, j] + \sum_{k=1}^{m-1} \Delta c[k, j] \cdot \prod_{r=1}^{k} y_{\pi_{rj}} \right\}. \tag{4}$$

Combining (3) and (4), the total cost of a solution $\mathbf{y}$ to the instance $[F|C]$ corresponding to an ordering matrix $\Pi \in perm(C)$ is given by the pseudo-Boolean polynomial

$$\begin{aligned} f_{[F|C],\Pi}(\mathbf{y}) &= \mathcal{F}_F(\mathbf{y}) + \mathcal{T}_{C,\Pi}(\mathbf{y}) \\ &= \sum_{i=1}^{m} f_i(1 - y_i) + \sum_{j=1}^{n} \left\{ \Delta c[0, j] + \sum_{k=1}^{m-1} \Delta c[k, j] \cdot \prod_{r=1}^{k} y_{\pi_{rj}} \right\}. \end{aligned} \tag{5}$$

It can be shown (Goldengorin *et al.* [21]) that the total cost function $f_{[F|C],\Pi}(\cdot)$ is identical for all $\Pi \in perm(C)$. We call this pseudo-Boolean polynomial the *Beresnev function* $\mathcal{B}_{[F|C]}(\mathbf{y})$ corresponding to the SPLP instance $[F|C]$ and $\Pi \in perm(C)$. In

other words

$$\mathcal{B}_{[F|C]}(\mathbf{y}) = f_{[F|C],\Pi}(\mathbf{y}) \text{ where } \Pi \in perm(C). \tag{6}$$

We can formulate (1) in terms of Beresnev functions as

$$\mathbf{y}^\star \in \arg\min\{\mathcal{B}_{[F|C]}(\mathbf{y}) : \mathbf{y} \in \{0,1\}^m, \mathbf{y} \neq \mathbf{1}\}. \tag{7}$$

Notice that if for clients p and q, $\{\pi_{1p}, \pi_{2p}, \ldots, \pi_{kp}\} = \{\pi_{1q}, \pi_{2q}, \ldots, \pi_{kq}\}$ for $k \leq n$, then the k^{th} order terms in the Beresnev function corresponding to these two clients can be aggregated. This implies that in general, the Beresnev function will be a more space-efficient representation of the SPLP than the conventional $[F|C]$ matrix. This representation also makes it easier to construct data structures that allow efficient updating operations in the algorithm presented below.

2.2 Fundamentals of Data Correcting

Data correcting is a method in which we alter the data in a problem instance to convert it to an instance that is easily solvable. This methodology was first introduced in Goldengorin [18]. In this subsection we illustrate the method for the SPLP when the instance data is represented by the fixed cost vector and the transportation cost matrix. However it can be applied to a wide variety of optimization problems, and in particular, to the SPLP represented as a Beresnev function.

Consider an instance $[F|C]$ of the SPLP. The objective of the problem is to compute a set P, $\emptyset \subset P \subseteq I$, that minimizes $f_{[F|C]}(P)$. Also consider a SPLP instance $[S|D]$ that is known to be polynomially solvable. Let $P^\star_{[F|C]}$ and $P^\star_{[S|D]}$ be optimal solutions to $[F|C]$ and $[S|D]$, respectively. Let us define the proximity measure $\rho([F|C], [S|D])$ between the two instances as

$$\rho([F|C], [S|D]) = \sum_{i \in I} |f_i - s_i| + \sum_{j \in J} \max\{|c_{ij} - d_{ij}| : i \in I\}. \tag{8}$$

We use $\max\{|c_{ij} - d_{ij}| : i \in I\}$ in (8) instead of the more intuitive $\sum_{i \in I}\{|c_{ij} - d_{ij}|\}$ since, in an optimal solution, the demand of each client is satisfied by a single facility, only one element in each column in the transportation matrix will contribute to the cost of the optimal solution.

Notice that $\rho([F|C], [S|D])$ is defined only when instances $[F|C]$ and $[S|D]$ are of the same size. Also note that it can be computed in time polynomial in the size of the two instances. The following theorem, which forms the basis of data correcting, shows

that $\rho([F|C], [S|D])$ is an upper bound to the difference between the *unknown* optimal costs for the SPLP instances $[F|C]$ and $[S|D]$.

Theorem 2.1 *Let $[F|C]$ and $[S|D]$ be two SPLP instances of the same size, and let $P^\star_{[F|C]}$ and $P^\star_{[S|D]}$ be optimal solutions to $[F|C]$ and $[S|D]$ respectively. Then*

$$|f_{[F|C]}(P^\star_{[F|C]}) - f_{[S|D]}(P^\star_{[S|D]})| \leq \rho([F|C], [S|D]).$$

PROOF. There are two cases to consider.
Case 1: $f_{[F|C]}(P^\star_{[F|C]}) \leq f_{[S|D]}(P^\star_{P[S|D]})$, and
Case 2: $f_{[F|C]}(P^\star_{[F|C]}) > f_{[S|D]}(P^\star_{[S|D]})$. We prove the Case 1 here. The proof of Case 2 is similar to the proof of Case 1.

$$\begin{aligned}
& f_{[F|C]}(P^\star_{[F|C]}) - f_{[S|D]}(P^\star_{[S|D]}) \\
&\leq \quad f_{[F|C]}(P^\star_{[S|D]}) - f_{[S|D]}(P^\star_{[S|D]}) \\
&= \quad \sum_{i \in P^\star_{[S|D]}} [f_i - s_i] + \sum_{j \in J} (\min\{c_{ij} : i \in P^\star_{[S|D]}\} - \min\{d_{ij} : i \in P^\star_{[S|D]}\}).
\end{aligned}$$

Let $c_{i_c(j)j} = \min\{c_{ij} : i \in P^\star_{[S|D]}\}$ and $d_{i_d(j)j} = \min\{d_{ij} : i \in P^\star_{[S|D]}\}$. Then

$$\begin{aligned}
& f_{[F|C]}(P^\star_{[F|C]}) - f_{[S|D]}(P^\star_{[S|D]}) \\
&\leq \quad \sum_{i \in P^\star_{[S|D]}} [f_i - s_i] + \sum_{j \in J} [c_{i_c(j)j} - d_{i_d(j)j}] \\
&\leq \quad \sum_{i \in P^\star_{[S|D]}} [f_i - s_i] + \sum_{j \in J} [c_{i_d(j)j} - d_{i_d(j)j}] \\
&\leq \quad \sum_{i \in P^\star_{[S|D]}} [f_i - s_i] + \sum_{j \in J} [\max\{c_{ij} - d_{ij} : i \in P^\star_{[S|D]}\}] \\
&\leq \quad \sum_{i \in P^\star_{[S|D]}} |f_i - s_i| + \sum_{j \in J} [\max\{|c_{ij} - d_{ij}| : i \in I\}] \\
&\leq \quad \sum_{i \in I} |f_i - s_i| + \sum_{j \in J} [\max\{|c_{ij} - d_{ij}| : i \in I\}] \\
&= \quad \rho([F|C], [S|D]).
\end{aligned}$$

■

Theorem 2.1 implies that if we have an optimal solution to a SPLP instance $[S|D]$, then we have an upper bound for *all* SPLP instances $[F|C]$ of the same size. This upper bound is actually the distance between the two instances, distances being defined by the accuracy measure (8). Also if the solution to $[S|D]$ can be computed in polynomial time, (i.e. $[S|D]$ belongs to a polynomially solvable special case) then an upper bound to the cost of an *as yet unknown* optimal solution to $[F|C]$ can be obtained in polynomial time. If the distance between the instances is not more than a prescribed accuracy α, then the optimal solution of $[S|D]$ is, in fact, a solution to $[F|C]$ within the prescribed accuracy. This theorem forms the basis of data correcting.

In general, the data correcting procedure works as follows. It assumes that we know a class of polynomially solvable instances of the problem. It starts by choosing a polynomially solvable SPLP instance $[S|D]$ from that class of instances, preferably as close as possible to the original instance $[F|C]$. If $\rho([F|C], [S|D]) \leq \alpha$, the procedure terminates and returns an optimal solution to $[S|D]$ as an approximation of an optimal solution to $[F|C]$. The instance $[F|C]$ is said to be 'corrected' to the instance $[S|D]$, which is solved polynomially to generate the solution output by the procedure. Otherwise, the set of feasible solutions for the problem is partitioned into two subsets. For the SPLP, one of these subsets comprises of solutions that locate a plant at a given site, and the other comprises of solutions that do not. The two new instances thus formed are perturbed in a way that they both change into instances that are within a distance α from a polynomially solvable instance. The procedure is continued until an instance with a proximity measure not more than α is obtained for all the subsets generated.

2.3 The Preliminary Preservation Procedure

The preliminary preservation procedure is one that tries to reduce the set of solutions in which to search for optimal solutions to a given instance. It applies to the minimization of supermodular (and maximization of submodular) functions. Since the objective function of the SPLP is supermodular, we can apply the procedure to this problem.

Consider the SPLP instance $[F|C]$, and subsets P_L and P_U of I, such that $\emptyset \subset P_L \subseteq P_U \subseteq I$. We can consider P_L as the set of sites where plants will definitely be located in an optimal solution, and P_U as the set of all candidate locations for locating plants in optimal solutions. In other words, plants will definitely *not* be located in any site belonging to $I \setminus P_U$ an optimal solution. Let $f^{\star}_{[F|C]}[P_L, P_U] = \min\{f_{[F|C]}(P) : P_L \subseteq P \subseteq P_U\}$. The following result is a straightforward application of Theorem 1 in Goldengorin *et al.* [20] to the SPLP.

Theorem 2.2 *Consider $P_L, P_U \subseteq I$, such that $\emptyset \subset P_L \subseteq P_U \subseteq I$, and let $k \in P_U \setminus P_L$. Then the following assertions hold:*

a. $f^{\star}_{[F|C]}[P_L, P_U \setminus \{k\}] - f^{\star}_{[F|C]}[P_L \cup \{k\}, P_U] \leq f_{[F|C]}(P_L) - f_{[F|C]}(P_L \cup \{k\})$;
b. $f^{\star}_{[F|C]}[P_L \cup \{k\}, P_U] - f^{\star}_{[F|C]}[P_L, P_U \setminus \{k\}] \leq f_{[F|C]}(P_U) - f_{[F|C]}(P_U \setminus \{k\})$.

An immediate corollary to Theorem 2.2 is the following.

Corollary 2.1 *Consider $P_L, P_U \subseteq I$, such that $\emptyset \subset P_L \subseteq P_U \subseteq I$, and let $k \in P_U \setminus P_L$. Then the following preservation rules are valid:*

Preservation Rule 1: If $f_{[F|C]}(P_U \setminus \{k\}) \geq f_{[F|C]}(P_U)$, then
$f^{\star}_{[F|C]}[P_L, P_U] = f^{\star}_{[F|C]}[P_L \cup \{k\}, P_U] \leq f^{\star}_{[F|C]}[P_L, P_U \setminus \{k\}]$.
Preservation Rule 2: If $f_{[F|C]}(P_L \cup \{k\}) \geq f_{[F|C]}(P_L)$, then
$f^{\star}_{[F|C]}[P_L, P_U] = f^{\star}_{[F|C]}[P_L, P_U \setminus \{k\}] \leq f^{\star}_{[F|C]}[P_L \cup \{k\}, P_U]$.

Using the preservation rules from Corollary 2.1, we can considerably reduce the search space for any given problem instance. This is done by the preliminary preservation procedure (PP) described in Figure 2.1. For a given instance $[F|C]$ of the SPLP, the procedure takes two sets P^i_L and P^i_U ($P^i_L \subseteq P^i_U$) as input, and outputs two sets P^o_L and P^o_U ($P^i_L \subseteq P^o_L \subseteq P^o_U \subseteq P^i_U$), such that $f^{\star}_{[F|C]}[P^i_L, P^i_U] = f^{\star}_{[F|C]}[P^o_L, P^o_U]$. The running time of the procedure is $\mathcal{O}(m^2)$ (Theorem 2, Goldengorin *et al.* [20]).

3. The Data Correcting Algorithm

The Data Correcting Algorithm (DCA) that we propose in this paper is one that uses a strong reduction procedure (RP, see Subsection 3.1) to reduce the original instance into a smaller 'core' instance, and then uses a data correcting procedure (DCP, see Subsection 3.2) to obtain a solution to the original instance, whose cost is not more than a pre-specified amount α more than the cost of an optimal solution.

3.1 The Reduction Procedure

The first preprocessing rules involving both fixed costs and transportation costs appeared in Khumawala [26]. In terms of Beresnev functions, these rules are stated as follows. We assume (without loss of generality) that we cannot partition I into sets I_1 and I_2, and J into sets J_1 and J_2, such that the transportation costs from sites in I_1 to

Procedure PP (P_L^i, P_U^i)
begin
$P_L^o \leftarrow P_L^i$; $P_U^o \leftarrow P_U^i$;

if $P_L^o = P_U^o$ **then**
return (P_U^o, P_U^o);

Compute $\delta^+ \leftarrow \min_{k \in P_U^o \setminus P_L^o}\{f_{[F|C]}(P_U^o) - f_{[F|C]}(P_U^o \setminus \{k\})\}$;
Compute $\delta^- \leftarrow \min_{k \in P_U^o \setminus P_L^o}\{f_{[F|C]}(P_L^o) - f_{[F|C]}(P_L^o \cup \{k\})\}$;

if $\delta^+ \leq 0$ **then** { *Preservation Rule 1* }
begin
Compute $r^+ \leftarrow \min\{k : f_{[F|C]}(P_U^o) - f_{[F|C]}(P_U^o \setminus \{k\}) = \delta^+\}$;
call PP($P_L^o \cup \{r^+\}$, P_U^o);
end
else if $\delta^- \leq 0$ **then** { *Preservation Rule 2* }
begin
Compute $r^- \leftarrow \min\{k : f_{[F|C]}(P_L^o) - f_{[F|C]}(P_L^o \cup \{k\}) = \delta^-\}$;
call PP(P_L^o, $P_U^o \setminus \{r^-\}$);
end
else return (P_L^o, P_U^o);
end;

Figure 2.1: The Preliminary Preservation Procedure

clients in J_2, and from sites in I_2 to clients in J_1 are not finite. We assume too, that the site indices are arranged in non-increasing order of $f_i + \sum_{j \in J} c_{ij}$ values.

Preprocessing Rule 1 *Let $\mathcal{B}_{[F|C]}(\mathbf{y})$ be the Beresnev function corresponding to the SPLP instance $[F|C]$ in which like terms have been aggregated. For each site index k, let a_k be the coefficient of the linear term corresponding to y_k and let t_k be the sum of the coefficients of all non-linear terms containing y_k. Then the following assertion holds.*

RO: *If $a_k \geq 0$, then there is an optimal solution $\mathbf{y}^\star$ in which $y_k^\star = 0$, else*
RC: *if $a_k + t_k \leq 0$, then there is an optimal solution $\mathbf{y}^\star$ in which $y_k^\star = 1$, provided that $y_i^\star \neq 1$ for some $i \neq k$.*

Notice that Preprocessing Rule 1 primarily tries to either open or close sites. If it succeeds, it also changes the Beresnev function for the instance, reducing the number of non-linear terms therein. In the remaining portion of this subsection, we describe a completely new reduction procedure (RP), whose primary aim is to reduce the coefficients of terms in the Beresnev function, and if we can reduce it to zero, to eliminate the term from the Beresnev function. This procedure is based on fathoming rules of branch and bound algorithms and data correcting principles.

Let us assume that we have an upper bound (UB) on the cost of an optimal solution for the given SPLP instance. This can be obtained by running a heuristic on the problem data. Now consider a non-linear term $\Delta c[k, j] \cdot \prod_{r=1}^{k} y_{\pi_{rj}}$ in the Beresnev function. This term will contribute to the cost of a solution, only if facilities are not located in sites $\pi_{1j}, \ldots, \pi_{kj}$. Let LB be a lower bound on the cost of solutions in which facilities are not located in sites $\pi_{1j}, \ldots, \pi_{kj}$. If $LB > UB$, then the non-linear term can be removed from the Beresnev function, thereby reducing the size of the Beresnev function. Note that a good upper bound yields a large reduction in the size of the Beresnev function. Otherwise if $LB \leq UB$, the coefficient of the term can be reduced by an amount $UB - LB - \varepsilon$, ($\varepsilon \geq 0$, small) without compromising the optimality of any optimal solution to the instance. Such changes in the Beresnev function alter the values of t_k, and can possibly allow us to use Preprocessing Rule 1 (assertion RC) to close certain sites. Once some sites are closed, some of the linear terms in the Beresnev function change into constant terms, and some of the quadratic terms change into linear ones. These changes cause changes in both the a_k and the t_k values, and can make further application of Preprocessing Rule 1 (both assertions RO and RC) possible, thus preprocessing some other sites, and making further changes in the Beresnev function. Using a good lower bound (for example, the bound in Erlenkotter [15]) it is often possible to discard a substantial portion of the nonlinear terms in the Beresnev function (see, for example, Table 4.2 in Section 4, in which more than 99% of the non-zero nonlinear terms are discarded). This reduction is time consuming (although polynomial), but in the preprocessing phase it can reduce large SPLP instances to much smaller core problems. A pseudocode of the reduction procedure (RP) is presented in Figure 3.1.

3.2 The Data Correcting Procedure

Let us suppose that the preliminary preservation procedure (PP) is applied to the SPLP instance $[F|C]$. On termination, it outputs two subsets P_L^o and P_U^o, $\emptyset \subset P_L^o \subseteq P_U^o \subseteq I$. If $P_L^o = P_U^o$, then the instance is said to have been solved by this procedure, and the set P_L^o is an optimal solution. Since the preliminary preservation procedure is a polynomial

Procedure RP ($\mathcal{B}_{[F|C]}(\mathbf{y})$)
begin
repeat
Compute an upper bound UB for the current subproblem;
for each nonlinear term $\Delta c[k, j] \cdot \prod_{r=1}^{k} y_{\pi_{rj}}$ in $\mathcal{B}_{[F|C]}(\mathbf{y})$ **do**
begin
Compute lower bound LB on the cost of solutions in which facilities are not located
in sites $\pi_{1j}, \ldots, \pi_{kj}$;
if $LB > UB$ **then**
Remove the term from $\mathcal{B}_{[F|C]}(\mathbf{y})$;
else
Reduce the coefficient of the term by $UB - LB - \varepsilon$;
end
Apply Preprocessing Rule 1 until no further sites could be preprocessed;
Recompute the Beresnev function $\mathcal{B}_{[F|C]}(\mathbf{y})$;
until no further preprocessing of sites was achieved in the current iteration;
end;

Figure 3.1: The Reduction Procedure

time algorithm, instances that it solves to optimality constitute a class of algorithmically defined polynomially solvable instances. We call such instances *PP-solvable*. We use this class of polynomially solvable instances in our algorithm, since it is one of the best among the polynomially solvable cases discussed in Goldengorin [19].

Next suppose that the given instance is not PP-solvable. In that case we try to extend the idea of the preliminary preservation procedure to obtain a solution, the difference between whose cost and the cost of the optimal solution is bounded by a pre-defined value α. This is the basic idea behind the data correcting procedure.

We will introduce a few notations to improve the readability of this subsection. Consider the SPLP instance $[F|C]$ and two sets P_L, $P_U \subseteq I$ such that $P_L \subset P_U$. Let

$$
\begin{aligned}
\delta_k^- &= f_{[F|C]}(P_L) - f_{[F|C]}(P_L \cup \{k\}) \text{ for each } k \in P_U \setminus P_L,\\
\delta^- &= \min\{\delta_k^- : k \in P_U \setminus P_L\}\\
r^- &= \min\{k : \delta_k^- = \delta^-,\ k \in P_U \setminus P_L\}\\
\delta_k^+ &= f_{[F|C]}(P_U) - f_{[F|C]}(P_U \setminus \{k\}) \text{ for each } k \in P_U \setminus P_L,\\
\delta^+ &= \min\{\delta_k^+ : k \in P_U \setminus P_L\}\\
r^+ &= \min\{k : \delta_k^+ = \delta^+,\ k \in P_U \setminus P_L\}.
\end{aligned}
$$

Note that $\delta_k^- = -a_k$ and $\delta_k^+ = a_k + t_k$ (see Goldengorin [19]). Therefore, these quantities can be calculated very efficiently when one uses a Beresnev function representation of the SPLP. Lemma 3.1 is a restatement of the preservation rules in Corollary 2.1.

Lemma 3.1 *Consider P_L, $P_U \subseteq I$, such that $\emptyset \subset P_L \subset P_U \subseteq I$, let $k \in P_U \setminus P_L$, and let P_A be an arbitrary subset of I. Then the following holds.*

a. If $\delta_k^- \leq 0$ and $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U \setminus \{k\}] \leq \gamma \leq \alpha$, then $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U] \leq \gamma \leq \alpha$; and

b. if $\delta_k^+ \leq 0$ and $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L \cup \{k\}, P_U] \leq \gamma \leq \alpha$, then $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U] \leq \gamma \leq \alpha$.

In case both δ^- and δ^+ are strictly positive, then Lemma 3.1 is no longer applicable, and the preliminary preservation procedure (PP) terminates. If $\delta = \min(\delta^-, \delta^+) \leq \alpha$, we could however correct the data of the instance so that either the costs of all solutions P, $P_L \subseteq P \subseteq P_U \setminus \{k\}$ increase by δ^-, or the costs of all solutions P, $P_L \cup \{k\} \subseteq P \subseteq P_U$ increase by δ^+, and Lemma 3.1 becomes applicable again. The solution that we hope to obtain by this correcting procedure will have an accuracy of δ according to the main theorem of data correcting (Theorem 2.1). Instead of changing the data in the instance, the same effect is achievable by decreasing the allowable accuracy value from α to $\alpha - \delta$. This gives rise to Lemma 3.2.

Lemma 3.2 *Consider P_L, $P_U \subseteq I$, such that $\emptyset \subset P_L \subset P_U \subseteq I$, let $k \in P_U \setminus P_L$, and let P_A be an arbitrary subset of I. Then the following holds.*

a. If $0 \leq \delta_k^- \leq \alpha$ and $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U \setminus \{k\}] \leq \gamma \leq \alpha - \delta_k^-$, then $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U] \leq \gamma + \delta_k^- \leq \alpha$; and

b. if $0 \leq \delta_k^+ \leq \alpha$ *and* $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L \cup \{k\}, P_U] \leq \gamma \leq \alpha - \delta_k^+$, *then* $f_{[F|C]}(P_A) - f^\star_{[F|C]}[P_L, P_U] \leq \gamma + \delta_k^+ \leq \alpha$.

PROOF. We prove the first part of the lemma. The proof of the second part is similar. There are two cases to consider.
Case 1: $f^\star_{[F|C]}[P_L, P_U] = f^\star_{[F|C]}[P_L, P_U \setminus \{k\}]$. In this case, the result follows trivially.
Case 2: $f^\star_{[F|C]}[P_L, P_U] = f^\star_{[F|C]}[P_L \cup \{k\}, P_U]$. From Theorem 2.2a,

$$f^\star_{[F|C]}[P_L, P_U \setminus \{k\}] - f^\star_{[F|C]}[P_L \cup \{k\}, P_U] \leq \delta_k^-$$
$$\iff f^\star_{[F|C]}[P_L, P_U \setminus \{k\}] \leq f^\star_{[F|C]}[P_L, P_U] + \delta_k^-.$$

The result follows. ∎

In case $\delta = \min(\delta^-, \delta^+) > \alpha$ then data correction cannot guarantee a solution within the prescribed allowable accuracy, and hence we need to use a branching procedure.

The data correcting procedure (DCP, see Figure 3.2) in our algorithm takes two sets $P_L, P_U \subseteq I$ ($\emptyset \subset P_L \subset P_U \subseteq I$) and α as input. It outputs a solution P^γ and a bound γ, such that $f_{[F|C]}(P^\gamma) - f_{[F|C]}(P^\star) \leq \gamma \leq \alpha$, where $P^\star$ is an optimal solution to $[F|C]$. It is a recursive procedure, that first tries to reduce the set $P_U \setminus P_L$ by applying Lemma 3.1. If Lemma 3.1 cannot be applied, then it tries to apply Lemma 3.2 to reduce it. We do not use the reduction procedure at this stage since it increases the computational times substantially without reducing the core problem appreciably. If even this lemma cannot be applied, then the procedure branches on a member $k \in P_U \setminus P_L$ and invokes two instances of DCP, one with sets $P_L \cup \{k\}$ and P_U, and the other with sets P_L and $P_U \setminus \{k\}$. Notice that the solutions searched by the two invocations of DCP are mutually exclusive and exhaustive. A bound is used to remove unpromising subproblems from the solution tree. The choice of the branching variable $k \in P_U \setminus P_L$ in DCP is motivated by the observation that $a_k < 0$ and $t_k + a_k > 0$ for each of these indices. (These are the preconditions of the branching rule.) A plant would have been located in this site in an optimal solution if the coefficient of linear term involving y_k in the Beresnev function would have been increased by $-a_k$. We could have predicted that a plant would not be located there if the same coefficient would have been decreased by $t_k + a_k$. Therefore we could use $\phi_k = \text{average}(-a_k, t_k + a_k) = \frac{t_k}{2}$ as a measure of the chance that we will *not* be able to predict the fate of site k in any subproblem of the current subproblem. If we want to reduce the size of the branch and bound tree

by assigning values to such variables, then we can think of a branching function that branches on the index $k \in P_U \setminus P_L$ with the largest ϕ_i value.

4. Computational Experiments

The execution of the DCA can be divided into two stages, a *preprocessing* stage in which the given instance is reduced to a core instance by using the RP; and a *solution* stage in which the core instance is solved using the DCP.

In the preprocessing stage we experimented with three procedures of reduction,

(a) using the "delta" and "omega" rules from Khumawala [26],
(b) using the RP with a combinatorial bound to obtain a lower bound, and
(c) using the RP with the Erlenkotter bound to obtain a lower bound.

The combinatorial bound (see Goldengorin *et al.* [20]) is a bound for general supermodular functions adapted for the SPLP.

We also experimented with the combinatorial bound and the Erlenkotter bound in the implementation of the DCP.

The effectiveness of the reduction procedure can be measured either by computing the number of free locations in the core instance, or by computing the number of non-zero nonlinear terms present in the Beresnev function of the core instance. Note that the number of non-zero nonlinear terms present in the Beresnev function is an upper bound on the number of unassigned customers in the core instance. Tables 4.1 and 4.2 shows how the various methods of reduction perform on the benchmark SPLP instances in the OR-Library [3]. The existing preprocessing rules due to Khumawala [26] and Goldengorin *et al.* [21] (i.e. procedure (a), which was used in the SPLP example in Goldengorin *et al.* [20]) cannot solve any of the OR-Library instances to optimality. However, the variants of the new reduction procedure (i.e. procedures (b) and (c)) solve a large number of these instances to optimality. Procedure (c), based on the Erlenkotter bound is marginally better than procedure (b) in terms of the number of free locations (Table 4.1), but substantially better in terms of the number of non-zero nonlinear terms in the Beresnev function (Table 4.2).

The information in Tables 4.1 and 4.2 can be combined to show that some of the problems that are not solved by these procedures can actually be solved by inspection of the core instances. For example, consider cap74. We see that the core problem (using

procedure (a)) has two free variables and one non-linear term. Therefore the Beresnev function of the core instance looks like

$$A + py_u + qy_w + ry_u y_w,$$

where $p, q < 0$, $r > 0$, $\min\{p + r, p + q\} > 0$ and A is a constant. The minima of such functions are easy to obtain by inspection.

In addition, Tables 4.1 and 4.2 demonstrate the superiority of the new preprocessing rule over the "delta" and "omega" rules. Consider for example the problem cap132. The "delta" and "omega" rules reduce the problem size from $m = 50$ and 2389 non-zero nonlinear variables to $m' = 27$ and 112 non-zero nonlinear variables. However, the new preprocessing rule reduces the same problem to one having $m' = 5$ and 3 non-zero nonlinear variables.

Table 4.1: Number of free locations after preprocessing SPLP instances in the OR-Library

Problem	m	n	Procedure		
			a	b	c
cap71	16	50	4	0	0
cap72	16	50	6	0	0
cap73	16	50	6	3	3
cap74	16	50	2	0	0
cap101	25	50	9	0	0
cap102	25	50	13	3	0
cap103	25	50	14	0	0
cap104	25	50	12	0	0
cap131	50	50	34	32	8
cap132	50	50	27	25	5
cap133	50	50	25	19	10
cap134	50	50	19	0	0

In order to test the effect of the bounds in the DCA, we compared the execution times of DCA using the two bounds on some difficult problems of the type suggested in Körkel [27] (see Subsection 4.4 for more details). The problems were divided into seven sets. Each set consists of five problems, each having 65 sites and 65 clients (see Subsection 4.4 for more details regarding these problems). From Table 4.3 we see that the Erlenkotter bound reduces the execution time taken by the combinatorial bound (that was used in the SPLP example in Goldengorin *et al.* [20]) by a factor more than

Table 4.2: Number of non-zero nonlinear terms in the Beresnev function after preprocessing SPLP instances in the OR-Library

Problem	Non-zero terms	Procedure		
	before preprocessing	a	b	c
cap71	699	6	0	0
cap72	699	12	0	0
cap73	699	13	2	2
cap74	699	1	0	0
cap101	1147	24	0	0
cap102	1147	33	2	0
cap103	1147	38	0	0
cap104	1147	29	0	0
cap131	2389	163	135	8
cap132	2389	112	92	3
cap133	2389	101	60	11
cap134	2389	62	0	0

100. This is not surprising, since the combinatorial bound is derived for a general supermodular function, while the Erlenkotter bound is specific to the SPLP.

Table 4.3: Comparison of bounds used with the DCA on Körkel-type instances with $m = n = 65$

Problem	Execution time of the DCP (sec)	
Set	Combinatorial Bound	Erlenkotter Bound
Set 1	119.078	0.022
Set 2	290.388	0.040
Set 3	458.370	0.056
Set 4	158.386	0.054
Set 9	428.598	0.588
Set10	542.530	0.998
Set11	479.092	2.280

We report our computational experience with the DCA on several benchmark instances of the SPLP in the remainder of this section. The performance of the algorithm is compared with that of the algorithms described in the papers that suggested these instances. We implemented the DCA in PASCAL, compiled it using Prospero Pascal, and ran it on a 733 MHz Pentium III machine.

4.1 Bilde and Krarup-type Instances

These are the earliest benchmark problems that we consider here. The exact instance data is not available, but the process of generating the problem instances is described in Bilde and Krarup [4]. There are 22 different classes of instances, and Table 4.4 summarizes their characteristics. In our experiments we generated 10 instances for each

Table 4.4: Description of the instances in Bilde and Krarup [4]

Problem Type	m	n	f_i	c_{ij}
B	50	100	Discrete Uniform (1000, 10000)	Discrete Uniform (0, 1000)
C	50	100	Discrete Uniform (1000, 2000)	Discrete Uniform (0, 1000)
Dq †	30	80	Identical, 1000×q	Discrete Uniform (0, 1000)
Eq †	50	100	Identical, 1000×q	Discrete Uniform (0, 1000)

† $q = 1, \ldots, 10$.

of the types of problems, and used the mean values of our solutions to evaluate the performance of our algorithm with the one used in Bilde and Krarup [4]. In our implementation, we used reduction procedure (b) and the combinatorial bound in the DCP.

The reduction procedure was not useful for these instances, but the DCA could solve all the instances in reasonable time. The results of our experiments are presented in Table 4.5. The performance of the algorithm implemented in Bilde and Krarup [4], was measured in terms of the number of branching operations performed by the algorithm and its execution time in CPU seconds on a IBM 7094 machine. We estimate the number of branching operations by our algorithm as the logarithm (to the base 2) of the number of subproblems it generated. From the table we see that the DCA reduces the number of subproblems generated by the algorithm in Bilde and Krarup [4] by a factor of 1000. This is especially interesting because Bilde and Krarup use a bound (discovered in 1967) identical to the Erlenkotter bound in their algorithm (see Körkel [27]) and we use the combinatorial bound. The CPU time required by the DCA to solve these problems were too low to warrant the use of any $\alpha > 0$.

4.2 Galvão and Raggi type Instances

Galvão and Raggi [16] developed a general 0-1 formulation of the SPLP and presented a 3-stage method to solve it. The benchmark instances suggested in this work are unique, in that the fixed costs are assumed to come from a Normal distribution rather than the more commonly used Uniform distribution. The transportation costs for an in-

Table 4.5: Results from Bilde and Krarup-type instances

	DCA		Branching	CPU time
Problem type	Avg. Branching	Avg. CPU time (sec)	reported in [4]	reported in [4]†
B	11.72	0.67	43.3	4.33
C	17.17	14.81	⋆	>250
D1	13.80	0.65	216	11
D2	12.13	0.38	218	24
D3	10.87	0.19	169	19
D4	10.25	0.15	141	17
D5	9.24	0.07	106	14
D6	8.99	0.09	101	15
D7	8.79	0.09	83	13
D8	8.60	0.09	55	11
D9	8.15	0.07	47	11
D10	7.29	0.03	43	11
E1	18.66	35.28	1271	202
E2	16.14	8.64	1112	172
E3	14.59	3.81	384	82
E4	13.65	2.74	258	65
E5	12.73	2.01	193	53
E6	11.82	0.90	136	43
E7	10.82	0.53	131	42
E8	10.79	0.68	143	48
E9	10.62	0.76	117	44
E10	10.36	0.69	79	37

† IBM 7094 seconds.

⋆ could not be solved in 250 seconds.

stance of size $m \times n$ with $m = n$ are computed as follows. A network, with a given arc density δ is first constructed, and the arcs in the network are assigned lengths sampled from a uniform distribution in the range $[1, n]$ (except for $n = 150$, where the range is $[1, 500]$). The transportation cost from i to j is the length of the cheapest path from i to j. The problem characteristics provided in Galvão and Raggi [16] are summarized in Table 4.6.

As with the data in Bilde and Krarup [4], the exact data for the instances are not known. So we generated 10 instances for each problem size, and used the mean values of the solutions for comparison purposes. In our DCA implementation, we used reduction procedure (b) and the combinatorial bound in the DCP. The comparative results are

Table 4.6: Description of the instances in Galvão and Raggi [16]

Problem Size	Density	Fixed costs' parameters	
$(m = n)$	δ	mean	standard deviation
10	0.300	4.3	2.3
20	0.150	9.4	4.8
30	0.100	13.9	7.4
50	0.061	25.1	14.1
70	0.043	42.3	20.7
100	0.025	51.7	28.9
150	0.018	186.1	101.5
200	0.015	149.5	94.4

given in Table 4.7. Since the computers used are different, we cannot make any comments on the relative performance of the solution procedures. However, since the average number of subproblems generated by the DCA is always less than 10 for each of these instances, we can conclude that these problems are easy for our algorithm. In fact they are too easy for the DCA to warrant $\alpha > 0$.

Table 4.7: Results from Galvão and Raggi-type instances

Problem	DCA					
Size	# solved by	Avg. # of	Avg. CPU	Avg. # of	CPU time	# of open plants
$(m = n)$	preprocessing	subproblems	time (sec)	open plants	reported in [16]†	reported in [16]
10	6	2.3	<0.001	4.7	<1	3
20	5	2.4	<0.001	9.0	<1	8
30	7	1.8	0.002	13.6	1	11
50	7	2.6	0.002	20.3	2	20
70	2	3.8	0.004	28.8	6	31
100	3	3.5	0.011	41.1	6	44
150	1	7.8	0.010	64.4	25	74
200	4	2.9	0.158	81.8	63	84

† IBM 4331 seconds.

Notice that the average number of opened plants in the optimal solutions to the instances we generated is quite close to the number of opened plants in the optimal solutions reported in Galvão and Raggi. Also notice that the reduction procedure was quite effective — it solved 35 of the 80 instances generated.

4.3 Instances from the OR-Library

The OR-Library [3] has a set of instances of the SPLP. These instances were solved in Beasley [2] using an algorithm based on the Lagrangian heuristic for the SPLP. Here too, we used reduction procedure (b) and the combinatorial bound in the DCP. We solved the problems to optimality using the DCA. The results of the computations are provided in Table 4.8. The execution times suggest that the DCA is faster than the Lagrangian heuristic described in Beasley [2]. The reduction procedure was also quite effective for these instances, solving 4 of the 16 instances to optimality, and reducing the number of free sites appreciably in the other instances. Once again the use of $\alpha > 0$ cannot be justified, considering the execution times of the DCA.

Table 4.8: Results from OR-Library instances

Problem			DCA			CPU time	# of
name	m	n	m after preprocessing	# of sub-problems generated	CPU time (sec)	reported in [2]†	open plants
cap71	16	50	⋆	0	<0.01	0.11	11
cap72	16	50	⋆	0	<0.01	0.08	9
cap73	16	50	⋆	0	<0.01	0.11	5
cap74	16	50	⋆	0	<0.01	0.05	4
cap101	25	50	9	6	<0.01	0.18	15
cap102	25	50	13	16	<0.01	0.16	11
cap103	25	50	14	16	<0.01	0.14	8
cap104	25	50	12	7	0.01	0.11	4
cap131	50	50	34	196	0.01	0.31	15
cap132	50	50	27	183	0.02	0.28	11
cap133	50	50	25	71	<0.01	0.29	8
cap134	50	50	19	25	<0.01	0.15	4

⋆ instance solved by preprocessing only.

† Cray-X-MP/28 seconds.

4.4 Körkel-type Instances with 65 sites

Körkel [27] described several relatively large Euclidean SPLP instances ($m = n = 100$, and $m = n = 400$) and used a branch and bound algorithm to solve these problems. The bound used is an improvement on a bound based on the dual of the linear programming relaxation of the SPLP due to Erlenkotter [15] and is extremely effective. The bound

due to Erlenkotter [15] is very effective because, for a large majority of SPLP instances, the optimal solution to the dual of the linear programming relaxation of the SPLP is integral. In this subsection, we use instances that have the same cost structure as the ones in Körkel [27] but for which $m = n = 65$. Instances of this size were not dealt with in Körkel [27]. We used reduction procedure (b) for the RP, and the combinatorial bound in the DCP.

In Körkel [27], 120 instances of each problem size are described. These can be divided into 28 sets (the first 18 sets contain 5 instances each, and the rest contain 3 instances each). We solved all the 120 instances we generated, and found out that the instances in Sets 1, 2, 3, 4, 10, 11, and 12 are more difficult to solve than others. We therefore used these instances in the experiments in this section. The transportation cost matrix for a Körkel instance of size $n \times n$ is generated by distributing n points in random within a rectangular area of size 700×1300 and calculating the Euclidean distances between them. The fixed cost are computed as in Table 4.9. The values of the results

Table 4.9: Description of the fixed costs for instances in Körkel [27]

Problem Set	# of instances	Fixed cost for i^{th} instance
Set 1	5	Identical, set at $141 + 6.6i$
Set 2	5	Identical, set at $174 + 6.6i$
Set 3	5	Identical, set at $207 + 6.6i$
Set 4	5	Identical, set at $174 + 66i$
Set10	5	Identical, set at $7170 + 660i$
Set11	5	Identical, set at $7120.5 + 333.3i$
Set12	5	Identical, set at $8787 + 333.3i$

that we present for each set is the average of the values obtained for all the instances in that set. Interestingly, the preprocessing rules were found to be totally ineffective for all of these problems. Since the fixed costs are identical for all the sites, the sites are distributed randomly over a region, and the variable cost matrix is symmetric, no site presents a distinct advantage over any other. This prevents our reduction procedure to open or close any site. Table 4.10 shows the variation in the costs of the solution output by the DCA with changes in α, and Table 4.11 shows the corresponding decrease in execution times.

Table 4.10: Costs of solutions output by the DCA on Körkel-type instances with 65 sites

Problem	Optimal	Acceptable accuracy⋆				
Set		1%	2%	3%	5%	10%
Set 1	6370.0	6404.8	6450.6	6480.6	6569.2	6781.0
Set 2	6920.6	6952.2	6971.4	7028.4	7123.8	7320.2
Set 3	7707.4	7738.0	7770.2	7797.6	7854.6	8053.8
Set 4	9601.2	9642.4	9680.2	9698.4	9786.6	9932.0
Set10	146691.2	146896.6	146909.6	147543.6	148062.0	151542.2
Set11	168598.4	168858.2	169655.0	170341.6	170597.0	173913.8
Set12	186386.3	186729.7	187112.0	188002.7	188854.2	192528.7

⋆ : As a percentage of the optimal cost.

Table 4.11: Execution times for the DCA on Körkel-type instances with 65 sites

Problem	Optimal	Acceptable accuracy⋆				
Set		1%	2%	3%	5%	10%
Set 1	119.078	90.948	70.758	55.494	43.200	20.426
Set 2	290.388	225.108	172.422	145.828	96.240	36.966
Set 3	458.370	339.420	259.022	203.036	150.216	50.378
Set 4	158.386	129.694	109.754	89.666	65.548	30.058
Set10	428.598	370.120	319.804	283.832	230.078	142.090
Set11	542.530	476.350	418.628	408.594	290.338	160.744
Set12	479.092	416.472	370.832	326.572	261.835	149.038

⋆ : As a percentage of the optimal cost.

The effect of varying the acceptable accuracy α on the cost of the solutions output by the DCA is also presented graphically in Figure 4.1. We define the *achieved accuracy* β as

$$\beta = \frac{\text{cost of solution output by the DCA} - \text{cost of an optimal solution}}{\text{cost of optimal solution}}$$

and the *relative time* τ as

$$\tau = \frac{\text{execution time for the DCA on the problem}}{\text{execution time for the DCA to compute an optimal solution for the problem}}$$

Note that the achieved accuracy β varies almost linearly with α, with a slope close to 0.5. Also note that the relative time τ of the DCA reduces with increasing α. The reduction is slightly better than linear, with an average slope of -8.

4.5 Körkel Instances with 100 Sites

We solved the benchmark instances in Körkel [27] with $m = n = 100$ to optimality and observed that the instances in Sets 10, 11, and 12 required relatively longer execution times. So we restricted further computations to instances in those sets. The fixed and transportation costs for these problems are computed in the procedure described in Subsection 4.4. Tables 4.12 and 4.13 show the results obtained by running the DCA on these problem instances. In our DCA implementation for solving these instances, we used reduction procedure (c) and the Erlenkotter bound in the DCP.

Table 4.12: Costs of solutions output by the DCA on Körkel-type instances with 100 sites

Problem	Optimal	Acceptable accuracy*				
Set		1%	2%	3%	5%	10%
Set10	190782.0	191550.8	192755.4	192080.6	195983.2	203934.2
Set11	219583.4	220438.8	222393.6	221947.2	228467.2	235963.4
Set12	240402.4	241609.6	243336.8	244209.4	247417.6	259168.6

* : As a percentage of the optimal cost.

Table 4.13: Execution times for the DCA on Körkel-type instances with 100 sites

Problem	Optimal	Acceptable accuracy*				
Set		1%	2%	3%	5%	10%
Set10	133.746	91.774	65.99	65.908	44.2	32.074
Set11	81.564	55.356	39.554	38.348	33.628	17.598
Set12	111.272	85.858	65.608	55.928	61.758	33.014

* : As a percentage of the optimal cost.

Figure 4.2 illustrates the effect of varying the acceptable accuracy α on the cost of the solutions output by the DCA for the instances mentioned above. The nature of the graphs is similar to those in Figure 4.1. However, in several of the instances we noticed that β reduced when α is increased, and in some other instances τ increased when α was increased.

5. Conclusions

In this paper we tailor the general data correcting algorithm (DCA) for supermodular functions (see Goldengorin *et al.* [20]) to the simple plant location problem (SPLP). This algorithm consists of two procedures, a reduction procedure to reduce the original instance to a smaller 'core' instance, and a data correcting procedure to solve the core instance.

Theorem 2.1 can be considered as the basis of data correcting. It states that for two different instances of the SPLP of the same size, the difference between the costs of the *unknown* optimal solutions for these instances is bounded by a polynomially calculated distance between these instances. This distance is used to *correct* the data of the original instance in an implicit way by just *correcting* the value of the acceptable accuracy parameter in the DCA.

An important contribution of this paper is a new reduction procedure, which when implemented in the DCA yields to a substantial reduction in the size of the original instance. This reduction procedure is much more powerful than the "delta" and "omega" reduction rules in Khumawala [26], which it uses as a component procedure. It also incorporates data correcting and the strong Erlenkotter bound specific to the SPLP (see Erlenkotter [15]), which is more computationally efficient than the combinatorial bound used in Goldengorin *et al.* [20]. The strength of the new reduction procedure based on the Erlenkotter bound is made obvious by the observation that none of the instances in the OR-Library [3] could be solved by the delta and omega rules to optimality, but the new reduction procedure solves 75% of them to optimality, and preprocesses at least twice the number of sites as the "delta" and "omega" rules for the remaining 25% of the instances. Another contribution of the paper is the incorporation of the Erlenkotter bound to the recursive branch and bound type data correcting procedure.

We have compared the performance of the Erlenkotter bound implemented in an usual branch-and-bound type algorithm (see Bilde and Krarup [4]) and the combinatorial bound implemented in the DCP for the new reduction rule and for fathoming subproblems created by the DCP. On the instances in Bilde and Krarup [4], the number of subproblems created by the branch-and-bound type algorithm with Erlenkotter bound is found to be more than 1000 times the number of subproblems created by the DCP based on the combinatorial bound.

We have tested the DCA on a broad range of different classes of instances available in the literature (Bilde and Krarup [4], Galvão and Raggi [16], OR-Library [3], Körkel [27]). The striking computational result is the ability of the DCA to find exact

solutions for many relatively large instances within fractions of second. For example, an exact global optimum of the 200×200 instances from Gãlvao and Raggi [16] was found within 0.2 seconds on a PC with a 733 MHz processor.

In all of our implementations for the DCA with combinatorial and Erlenkotter bounds we have used data structures induced by pseudo-Boolean representation of the SPLP due to Beresnev [5]. These data structures are conducive to efficient updating for the current subproblems in the DCA and sometimes show that a current subproblem remaining after application of the new reduction procedure has relatively small number of linear and non-linear terms in the corresponding Beresnev function and therefore can be solved by any branch and bound type algorithm for the SPLP.

We have found that for all instances in Körkel [27] the "delta" and "omega" reduction rules were totally ineffective since none of the sites present any distinct advantage over any other (the fixed costs are almost identical for all sites, the sites are distributed randomly over a region, and the transportation costs matrix is symmetric). Anyway, the DCA has solved to optimality all the instances with $m = n = 100$ within fractions of a second except the instances in Sets 10, 11 and 12 which required relatively longer execution times. On these sets of instances we have studied the behavior of the execution time and calculated accuracy depending on the values of the acceptable accuracy α. When the acceptable accuracy parameter α increases, we see that (with a few exceptions) the costs of the solutions output by the DCA worsen, but the execution times also decrease. At $\alpha = 10\%$ of the optimal solution cost, the cost of the solution output by the DCA is 6% to 8% suboptimal, but the execution time reduces to less than 20% of that required to obtain the optimal solution. The increase in the solution costs is almost linear with α, and the rate of decrease in the solution time decreases appreciably when $\alpha > 0.3$.

Our computational experience with the DCA on several benchmark instances known in the literature suggests that the algorithm compares well with other algorithms known for the problem. However, the DCA, like any other branch and bound algorithm, depends heavily on the quality of the bounds used. In summary, our computations suggest that the DCA described in this paper is a competitive algorithm to solve SPLP instances.

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Procedure DCP (P_L, P_U, α)
begin
if $P_L = P_U$ **then**
return $(P_U, 0)$;

Compute $\delta^+, \delta^-, r^+, r^-$;

{ *Apply Lemma 3.1 (Preliminary Preservation)* }
if $\delta^+ \leq 0$ **then** { *Lemma 3.1(b)* }
begin
$(P^\gamma, \gamma) \leftarrow \text{DCP}(P_L \cup \{r^+\}, P_U, \alpha)$;
end
else if $\delta^- \leq 0$ **then** { *Lemma 3.1(a)* }
begin
$(P^\gamma, \gamma) \leftarrow \text{DCP}(P_L, P_U \setminus \{r^-\}, \alpha)$;
end
{ *Apply Lemma 3.2 (Data Correction)* }
else if $\delta^+ \leq \alpha$ **then** { *Lemma 3.2(b)* }
begin
$(P^\gamma, \gamma) \leftarrow \text{DCP}(P_L \cup \{r^+\}, P_U, \alpha - \delta^+)$;
$\gamma \leftarrow \delta^+$;
end
else if $\delta^- \leq \alpha$ **then** { *Lemma 3.2(a)* }
begin
$(P^\gamma, \gamma) \leftarrow \text{DCP}(P_L, P_U \setminus \{r^-\}, \alpha - \delta^-)$;
$\gamma \leftarrow \delta^-$;
end
{ *Branch* }
else
begin
select $k \in P_U^o \setminus P_L^o$; { *Branching Rule* }
if the bound obtained using the sets $P_L \cup \{k\}$ and P_U is better than
the best solution found so far, **then**
$(P^{\gamma^+}, \gamma^+) \leftarrow \text{DCP}(P_L \cup \{k\}, P_U, \alpha)$;
if the bound obtained using the sets P_L and $P_U \setminus \{k\}$ is better than
the best solution found so far, **then**
$(P^{\gamma^-}, \gamma^-) \leftarrow \text{DCP}(P_L, P_U \setminus \{k\}, \alpha)$;
$P^\gamma \leftarrow \arg\min\{f_{[F|C]}(P^{\gamma^+}), f_{[F|C]}(P^{\gamma^-})\}$;
$\gamma \leftarrow \min\{f_{[F|C]}(P^{\gamma^+}), f_{[F|C]}(P^{\gamma^-})\} - \min\{f_{[F|C]}(P^{\gamma^+}) - \gamma^+, f_{[F|C]}(P^{\gamma^-}) - \gamma^-\}$;
end

return (P^γ, γ);
end;

Figure 3.2: The Data Correcting Procedure

Figure 4.1: Performance of the DCA for Körkel-type instances with 65 sites

Figure 4.2: Performance of the DCA for Körkel-type instances with 100 sites